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Fixed-Order Linear Parameter-Varying Feedback Control of a Lab-Scale Overhead Crane

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***Abstract*—This paper presents a numerically attractive approach to design fixed-order $\mathcal{H}_2/\mathcal{H}_\infty$ controllers for discrete-time linear parameter-varying (LPV) systems. In this approach, the controller order, which is completely determined by the number of states and the parameter-dependency, is selected in advance. For a prefixed controller order, parameter-dependent sufficient linear matrix inequalities (LMIs) are presented, relying on an a priori computed full-order LPV controller that stabilizes the LPV system for all possible parameter trajectories. Pólya's theorem and polynomial approximations are used to obtain numerically tractable LMI problems that guarantee feasibility of the parameter-dependent synthesis conditions. The practical viability of the approach is demonstrated by experimental validations on a lab-scale overhead crane with varying cable length.**

***Index Terms*—linear matrix inequalities, H-infinity control, linear feedback control systems, linear parameter-varying systems, output feedback**

I. Introduction

BRIDGING the gap between the restricted class of linear time-invariant (LTI) systems and the general class of nonlinear systems, the modern framework of linear parameter-varying (LPV) systems has gained popularity since the nineties. It has been successful in many applications ranging from wafer stages [1] to racing motorcycles [2]. See the recent survey [3] and book [4] for a complete overview.

Various effective solutions relying on convex optimization have been proposed for full-order LPV control [5]–[10]. On the other hand, the design of *fixed-order* LPV controllers (i.e., with a prefixed number of states and parameter dependency) has not yet been extensively studied and applied. Namely, even for LTI systems, the fixed-order control design problem is nonconvex and thus hard to solve. Various fixed-order synthesis approaches have been proposed, relying on either directly addressing the nonconvex problem or on the derivation of convex sufficient conditions. For instance, nonsmooth optimization techniques have been employed to design reduced-order controllers for LTI systems [11], [12], but cannot handle LPV dynamics. At the same time, sequential convex programming methods, see [13]–[15], can cope with LPV dynamics, but are computationally demanding. Alternatively, the convex approach [16] relies on the design of numerous random initial controllers, which are subsequently used for the design of a single LPV controller, resulting in a numerically costly procedure. Moreover, the latter approach does not allow all system matrices to be parameter-dependent. Although the two-step convex approach presented in [17] (using a stabilizing state feedback for a specific augmented system as a starting point) is extendable to handle LPV dynamics, no guidelines are provided on how to select an initial state feedback to obtain a high performance reduced-order controller. The latter approach is inspired on the idea originally proposed in [18]–[20].

In this paper, the recently developed convex approach [21] for the design of reduced-order LTI controllers is extended to handle discrete-time LPV dynamics. The resulting approach considers the design of fixed-order LPV controllers with a rational parameter dependency where the polynomial degree of the numerator and denominator as well as the number of states is prefixed. A precomputed full-order LPV controller is required, which is selected according to intuitive guidelines. Pólya relaxations and polynomial approximations are used to obtain tractable LMI formulations whose feasibility guarantees feasibility of the parameter-dependent synthesis conditions. The combination of all above properties makes our method attractive, both computationally and practically, compared to existing approaches. Its merits are illustrated by experimental validations on an overhead crane test setup with varying cable length, comparing the experimental performances of a full-order controller with a significantly simpler fixed-order LPV controller. Compared to the preliminary results presented in [22], this paper provides both $\mathcal{H}_2$ and $\mathcal{H}_\infty$ synthesis conditions and completed experimental validations. An extension of the proposed approach with an iterative procedure to gradually reduce conservatism in a fixed-order controller design is presented in [23].

The paper is organized as follows. First, Section II discusses the mathematical problem formulation. Then, the fixed-order LPV controller design approach is presented in Section III, followed by numerical and experimental validations in Section IV. Finally, the conclusions are provided in Section V.

Notation: The set of nonnegative (positive) integers is denoted by $\mathbb{N}$ ($\mathbb{N}_+$), while I_n denotes the identity matrix of dimension n and $0_{m\times n}$ is a zero matrix of dimension $m \times n$. The subscripts are omitted when the dimensions can be inferred from the context. The transpose of a matrix X is written as X', and the notation $\mathrm{He}\{X\} = X + X'$ is

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used to shorten formulas. $X_\perp$ is an arbitrary matrix whose colums form a basis for the nullspace of X. The sets of real symmetric (positive definite) matrices of dimension n are denoted by $\mathbb{S}^n$ ($\mathbb{S}^n_+$). A star ($\star$) indicates symmetric terms in matrix inequalities.

II. Problem Formulation

We consider the discrete-time LPV state-space model

$$\begin{cases} x(k+1) &= A(\alpha(k))x(k) + B_w(\alpha(k))w(k) \\ & \quad +B_u(\alpha(k))u(k), \\ z(k) &= C_z(\alpha(k))x(k) + D_w(\alpha(k))w(k) \\ & \quad +D_u(\alpha(k))u(k), \\ y(k) &= C_y(\alpha(k))x(k) + D_y(\alpha(k))w(k), \end{cases} \tag{1}$$

$k \in \mathbb{N}$, with state $x(k) \in \mathbb{R}^{n_x}$, exogenous input $w(k) \in \mathbb{R}^{n_w}$, control input $u(k) \in \mathbb{R}^{n_u}$, regulated output $z(k) \in \mathbb{R}^{n_z}$ and measured output $y(k) \in \mathbb{R}^{n_y}$. It is assumed that all system matrices are bounded for $k \in \mathbb{N}$ and have a polynomial dependency on the time-varying parameter $\alpha : \mathbb{N} \to \mathbb{R}^N$, $N \in \mathbb{N}_+$. The parameter rate of variation (at time k) is defined as $\Delta\alpha(k) := \alpha(k+1) - \alpha(k)$. Accordingly, the set of admissible parameter trajectories is given by

$$\mathcal{T} := \left\{\alpha : \mathbb{N} \to \mathbb{R}^N \middle| (\alpha(k), \Delta\alpha(k)) \in \Omega,\ \forall k \in \mathbb{N}\right\}, \tag{2}$$

where $\Omega \subset \mathbb{R}^{2N}$ is assumed to be a bounded convex polytope. Hence, both the parameter and its rate of variation are bounded.

The objective is to design a fixed-order LPV controller

$$K : \begin{cases} x_c(k+1) &= A_c(\alpha(k))x_c(k) + B_c(\alpha(k))y(k), \\ u(k) &= C_c(\alpha(k))x_c(k) + D_c(\alpha(k))y(k), \end{cases} \tag{3}$$

stabilizing the LPV system (1) and satisfying multiple closed-loop performance specifications for all $\alpha \in \mathcal{T}$. The controller order is a priori fixed by specifying the state dimension and parameter dependency. Specifically, we consider the design of reduced-order ($x_c(k) \in \mathbb{R}^q$, $q < n_x$) LPV controllers (3) with a rational parameter dependency, where the polynomial degrees of the numerator and denominator are preselected.

Grouping the controller matrices of (3) as

$$\Theta(\alpha) := \begin{bmatrix} A_c(\alpha) & B_c(\alpha) \\ C_c(\alpha) & D_c(\alpha) \end{bmatrix}, \tag{4}$$

the closed-loop interconnection of the LPV system (1) with an LPV controller (3) is indicated using a subscript as

$$H_\Theta : \begin{cases} x_{cl}(k+1) &= \mathcal{A}_\Theta(\alpha(k))x_{cl}(k) + \mathcal{B}_\Theta(\alpha(k))w(k), \\ z(k) &= \mathcal{C}_\Theta(\alpha(k))x_{cl}(k) + \mathcal{D}_\Theta(\alpha(k))w(k), \end{cases} \tag{5}$$

where $x_{cl}(k) = \begin{bmatrix} x(k)' & x_c(k)' \end{bmatrix}' \in \mathbb{R}^{n_x+q}$ is a closed-loop state vector. Defining the matrices

$$\begin{bmatrix} \tilde{A}(\alpha) & \tilde{B}_w(\alpha) & \tilde{B}_u(\alpha) \\ \tilde{C}_z(\alpha) & \tilde{D}_w(\alpha) & \tilde{D}_u(\alpha) \\ \tilde{C}_y(\alpha) & \tilde{D}_y(\alpha) & 0 \end{bmatrix} := \left[\begin{array}{cc|c|cc} A(\alpha) & 0 & B_w(\alpha) & 0 & B_u(\alpha) \\ 0 & 0 & 0 & I_q & 0 \\ \hline C_z(\alpha) & 0 & D_w(\alpha) & 0 & D_u(\alpha) \\ \hline 0 & I_q & 0 & 0 & 0 \\ C_y(\alpha) & 0 & D_y(\alpha) & 0 & 0 \end{array}\right], \tag{6}$$

the affine dependency of the closed-loop matrices in (5) on $\Theta(\alpha)$ is expressed as

$$\begin{bmatrix} \mathcal{A}_\Theta(\alpha) & \mathcal{B}_\Theta(\alpha) \\ \mathcal{C}_\Theta(\alpha) & \mathcal{D}_\Theta(\alpha) \end{bmatrix} = \begin{bmatrix} \tilde{A}(\alpha) & \tilde{B}_w(\alpha) \\ \tilde{C}_z(\alpha) & \tilde{D}_w(\alpha) \end{bmatrix} + \begin{bmatrix} \tilde{B}_u(\alpha) \\ \tilde{D}_u(\alpha) \end{bmatrix} \Theta(\alpha) \begin{bmatrix} \tilde{C}_y(\alpha) & \tilde{D}_y(\alpha) \end{bmatrix}. \tag{7}$$

III. Fixed-Order LPV Controller Design

This section presents convex parameter-dependent sufficient conditions to design fixed-order LPV controllers of the form (3) for the LPV system (1), such that the closed-loop system is exponentially stable for all parameter trajectories $\alpha \in \mathcal{T}$, and, moreover, satisfies multiple closed-loop $\mathcal{H}_2$ and/or $\mathcal{H}_\infty$ performance specifications. These conditions rely on a polynomially parameter-dependent full-order $\mathcal{H}_2/\mathcal{H}_\infty$ LPV controller for the same system, which can be computed using, for instance, the convex approaches discussed in [6]–[8].

Defining $\mathcal{H}_2$ performance similarly as in [24]–[26], and $\mathcal{H}_\infty$ performance as in [8], extended parameter-dependent LMI characterizations for $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance are provided in Subsection III-A. They form the starting point for the derivation of the synthesis conditions for fixed-order $\mathcal{H}_2$ and $\mathcal{H}_\infty$ LPV controller design presented in Subsection III-B. The latter conditions are straightforwardly adapted to handle multi-objective control problems. Finally, Subsection III-C briefly describes how to obtain numerically tractable LMI formulations.

A. $\mathcal{H}_2$ and $\mathcal{H}_\infty$ analysis conditions

This subsection presents extended parameter-dependent LMIs for $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance analysis of the discrete-time LPV system (1) in closed-loop with a given controller $\Theta(\alpha)$, as defined in (4). In these parameter-dependent LMIs, the closed-loop performance of $\Theta(\alpha)$ is linked to a full-order polynomially parameter-dependent controller, represented by a matrix $\Psi(\alpha)$ with dimension $(n_x + n_u) \times (n_x + n_y)$, as in (4). We augment $\Theta(\alpha)$ with Schur stable unobservable and/or uncontrollable dynamics to form a so-called lifted controller matrix $\Theta_a(\alpha)$ with the same dimensions as $\Psi(\alpha)$. Since H_Θ and H_{Θ_a} (the interconnection of (1) with the controller Θ_a) share the same stability and performance properties, the lifted controller $\Theta_a(\alpha)$ allows us to characterize stability and performance of H_Θ in terms of H_Ψ. Namely, defining

$$\Upsilon(\alpha) := \Theta_a(\alpha) - \Psi(\alpha),$$

note that

$$\begin{bmatrix} \mathcal{A}_{\Theta_a}(\alpha) & \mathcal{B}_{\Theta_a}(\alpha) \\ \mathcal{C}_{\Theta_a}(\alpha) & \mathcal{D}_{\Theta_a}(\alpha) \end{bmatrix} = \begin{bmatrix} \mathcal{A}_\Psi(\alpha) & \mathcal{B}_\Psi(\alpha) \\ \mathcal{C}_\Psi(\alpha) & \mathcal{D}_\Psi(\alpha) \end{bmatrix} + \begin{bmatrix} \tilde{B}_u(\alpha) \\ \tilde{D}_u(\alpha) \end{bmatrix} \Upsilon(\alpha) \begin{bmatrix} \tilde{C}_y(\alpha) & \tilde{D}_y(\alpha) \end{bmatrix}. \tag{8}$$

In this paper, we select a Kalman canonical form for the lifted controller matrix

$$\Theta_a(\alpha) = \left[\begin{array}{cc|c} A_c(\alpha) & A_{12}(\alpha) & B_c(\alpha) \\ 0 & A_{22}(\alpha) & 0 \\ \hline C_c(\alpha) & C_2(\alpha) & D_c(\alpha) \end{array}\right], \tag{9}$$

where $A_{22}(\alpha)$ is Schur stable for all $\alpha \in \mathcal{T}$.

Based on relation (8), extended parameter-dependent LMI characterizations for $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance of the discrete-time LPV system (5) are presented in the following theorems. In order to facilitate the presentation of these characterizations, we define the following parameter-dependent matrices

$$Q_2(\alpha, \Psi(\alpha)) := \left[\begin{array}{cc|c} I & 0 & 0 \\ \mathcal{A}_\Psi(\alpha) & \mathcal{B}_\Psi(\alpha) & \tilde{B}_u(\alpha) \\ 0 & I & 0 \end{array}\right],$$

$$Q_\infty(\alpha, \Psi(\alpha)) := \left[\begin{array}{cc|c} I & 0 & 0 \\ \mathcal{A}_\Psi(\alpha) & \mathcal{B}_\Psi(\alpha) & \tilde{B}_u(\alpha) \\ 0 & I & 0 \\ \mathcal{C}_\Psi(\alpha) & \mathcal{D}_\Psi(\alpha) & \tilde{D}_u(\alpha) \end{array}\right].$$

Theorem 1 (Extended $\mathcal{H}_2$ performance)**.** *Let $\Psi(\alpha) \in \mathbb{R}^{(n_x+n_u)\times(n_x+n_y)}$ be an arbitrary bounded matrix for all $(\alpha, \Delta\alpha) \in \Omega$, and let $\Theta_{\mathrm{a}}(\alpha) \in \mathbb{R}^{(n_x+n_u)\times(n_x+n_y)}$ be constructed from $\Theta(\alpha) \in \mathbb{R}^{(q+n_u)\times(q+n_y)}$ by adding Schur stable uncontrollable and/or unobservable dynamics. Then, the closed-loop system H_Θ, defined as in* (5)*, is exponentially stable and $\|H_\Theta\|_2^2 < \mu$ if there exist bounded matrices $P(\alpha) \in \mathbb{S}_+^{2n_x}$, $W(\alpha) \in \mathbb{S}^{n_z}$, $X_1(\alpha) \in \mathbb{R}^{2n_x\times(n_x+n_u)}$, $X_2(\alpha) \in \mathbb{R}^{n_w\times(n_x+n_u)}$, $X_3(\alpha) \in \mathbb{R}^{(n_x+n_u)\times(n_x+n_u)}$, $X_4(\alpha) \in \mathbb{R}^{n_z\times(n_x+n_u)}$, $X_5(\alpha) \in \mathbb{R}^{2n_x\times(n_x+n_u)}$, $X_6(\alpha) \in \mathbb{R}^{n_w\times(n_x+n_u)}$ and $X_7(\alpha) \in \mathbb{R}^{(n_x+n_u)\times(n_x+n_u)}$, for all $(\alpha, \Delta\alpha) \in \Omega$, such that* $\mathrm{Trace}\{W(\alpha)\} < \mu$ *and the parameter-dependent LMIs* (10a) *and* (10b) *hold for all $(\alpha, \Delta\alpha) \in \Omega$.*

Proof. The proof, which is based on application of the projection lemma [27], is an extension of the proof for LTI systems presented in [21]. □

Theorem 2 (Extended $\mathcal{H}_\infty$ performance)**.** *Let $\Psi(\alpha) \in \mathbb{R}^{(n_x+n_u)\times(n_x+n_y)}$ be an arbitrary bounded matrix for all $(\alpha, \Delta\alpha) \in \Omega$, and let $\Theta_{\mathrm{a}}(\alpha) \in \mathbb{R}^{(n_x+n_u)\times(n_x+n_y)}$ be constructed from $\Theta(\alpha) \in \mathbb{R}^{(q+n_u)\times(q+n_y)}$ by adding Schur stable uncontrollable and/or unobservable dynamics. Then, the closed-loop system H_Θ, defined as in* (5)*, is exponentially stable and $\|H_\Theta\|_\infty^2 < \gamma$ if there exist bounded matrices $P(\alpha) \in \mathbb{S}_+^{2n_x}$, $X_1(\alpha) \in \mathbb{R}^{2n_x\times(n_x+n_u)}$, $X_2(\alpha) \in \mathbb{R}^{n_w\times(n_x+n_u)}$ and $X_3(\alpha) \in \mathbb{R}^{(n_x+n_u)\times(n_x+n_u)}$, for all $(\alpha, \Delta\alpha) \in \Omega$, such that the parameter-dependent LMI* (11) *holds for all $(\alpha, \Delta\alpha) \in \Omega$.*

Proof. The proof is an extension of the proof for LTI systems presented in [21]. □

In summary, Theorem 1 (Theorem 2) provides sufficient parameter-dependent LMIs for $\mathcal{H}_2$ ($\mathcal{H}_\infty$) performance analysis of the LPV system (1) in closed-loop with a given fixed-order LPV controller (3). Note that elimination of the slack variables $X_j(\alpha)$ in the LMIs (10) (LMIs (11)) yields well-known equivalent $\mathcal{H}_2$ ($\mathcal{H}_\infty$) analysis conditions for $H_{\Theta_{\mathrm{a}}}$ (and hence for H_Θ), see e.g. [24], [28], [29]. While the choice of $\Psi(\alpha)$ is irrelevant in the analysis conditions (10) and (11), the synthesis conditions that are presented in the next subsection require a *stabilizing* controller $\Psi(\alpha)$, see the discussion below Remark 1.

B. $\mathcal{H}_2$ and $\mathcal{H}_\infty$ synthesis conditions

This subsection presents parameter-dependent LMI conditions for the design of fixed-order LPV controllers of the form (3) for the discrete-time LPV system (1), such that an upper bound on the closed-loop $\mathcal{H}_2$ or $\mathcal{H}_\infty$ performance is guaranteed for all parameter trajectories $\alpha \in \mathcal{T}$.

Theorem 3 (Fixed-order $\mathcal{H}_2$ LPV synthesis)**.** *Let $\Psi(\alpha) \in \mathbb{R}^{(n_x+n_u)\times(n_x+n_y)}$ parameterize a stabilizing full-order controller for the LPV system* (1)*, and let $\mathcal{A}_\Psi(\alpha)$, $\mathcal{B}_\Psi(\alpha)$, $\mathcal{C}_\Psi(\alpha)$ and $\mathcal{D}_\Psi(\alpha)$ denote the corresponding closed-loop matrices, as in* (5)*. For a predefined controller order q ($0 \le q < n_x$), let $A_{22}(\alpha) \in \mathbb{R}^{(n_x-q)\times(n_x-q)}$ be a given matrix that is Schur stable for all $(\alpha, \Delta\alpha) \in \Omega$. If there exist bounded matrices $P(\alpha) \in \mathbb{S}_+^{2n_x}$, $W(\alpha) \in \mathbb{S}^{n_z}$,*

$$\bar{\Theta}(\alpha) = \begin{bmatrix} \bar{\Theta}_{11}(\alpha) & \bar{\Theta}_{12}(\alpha) & \bar{\Theta}_{13}(\alpha) \\ 0 & 0_{(n_x-q)\times(n_x-q)} & 0 \\ \bar{\Theta}_{21}(\alpha) & \bar{\Theta}_{22}(\alpha) & \bar{\Theta}_{23}(\alpha) \end{bmatrix} \tag{12}$$

with $\bar{\Theta}_{11}(\alpha) \in \mathbb{R}^{q\times q}$, $\bar{\Theta}_{12}(\alpha) \in \mathbb{R}^{q\times(n_x-q)}$ and $\bar{\Theta}_{23}(\alpha) \in \mathbb{R}^{n_u\times n_y}$, and

$$Y(\alpha) = \begin{bmatrix} Y_{11}(\alpha) & Y_{12}(\alpha) & Y_{13}(\alpha) \\ 0 & Y_{22}(\alpha) & 0 \\ Y_{31}(\alpha) & Y_{32}(\alpha) & Y_{33}(\alpha) \end{bmatrix} \tag{13}$$

with $Y_{11}(\alpha) \in \mathbb{R}^{q\times q}$, $Y_{22}(\alpha) \in \mathbb{R}^{(n_x-q)\times(n_x-q)}$, and $Y_{33}(\alpha) \in \mathbb{R}^{n_u\times n_u}$, for all $(\alpha, \Delta\alpha) \in \Omega$, and a scalar μ such that $\mathrm{Trace}\{W(\alpha)\} < \mu$ *and the parameter-dependent LMIs* (14) *hold for all $(\alpha, \Delta\alpha) \in \Omega$, where $Z(\alpha)$ is given by*

$$\begin{aligned} Z(\alpha) &:= \bar{\Theta}(\alpha) \\ &\quad + Y(\alpha)\left(\begin{bmatrix} 0_{q\times q} & 0 & 0 \\ 0 & A_{22}(\alpha) & 0 \\ 0 & 0 & 0_{n_u\times n_y} \end{bmatrix} - \Psi(\alpha)\right), \end{aligned} \tag{15}$$

then the fixed-order LPV controller parameterized by

$$\Theta(\alpha) = \begin{bmatrix} Y_{11}(\alpha) & Y_{13}(\alpha) \\ Y_{31}(\alpha) & Y_{33}(\alpha) \end{bmatrix}^{-1} \begin{bmatrix} \bar{\Theta}_{11}(\alpha) & \bar{\Theta}_{13}(\alpha) \\ \bar{\Theta}_{21}(\alpha) & \bar{\Theta}_{23}(\alpha) \end{bmatrix} \tag{16}$$

stabilizes the closed-loop system (5) *with a guaranteed upper bound $\sqrt{\mu}$ on its $\mathcal{H}_2$ performance.*

Proof. The proof is an extension of the proof for LTI systems presented in [21], to the case of parameter-dependent matrices. □

Theorem 4 (Fixed-order $\mathcal{H}_\infty$ LPV synthesis)**.** *Let $\Psi(\alpha) \in \mathbb{R}^{(n_x+n_u)\times(n_x+n_y)}$ parameterize a stabilizing full-order controller for the LPV system* (1)*, and let $\mathcal{A}_\Psi(\alpha)$, $\mathcal{B}_\Psi(\alpha)$, $\mathcal{C}_\Psi(\alpha)$ and $\mathcal{D}_\Psi(\alpha)$ denote the corresponding closed-loop matrices, as in* (5)*. For a predefined controller order q ($0 \le q < n_x$), let $A_{22}(\alpha) \in \mathbb{R}^{(n_x-q)\times(n_x-q)}$ be a given matrix that is Schur stable for all $(\alpha, \Delta\alpha) \in \Omega$. If there exist bounded matrices $P(\alpha) \in \mathbb{S}_+^{2n_x}$, $\bar{\Theta}(\alpha) \in \mathbb{R}^{(q+n_u)\times(n_x+n_y)}$ as in* (12)*, and $Y(\alpha)$ as in* (13)*, for all $(\alpha, \Delta\alpha) \in \Omega$, and a scalar γ such that the parameter-dependent LMI* (17) *holds for all $(\alpha, \Delta\alpha) \in \Omega$, where $Z(\alpha)$ is given by* (15)*, then the fixed-order LPV controller parameterized by* (16) *stabilizes the closed-loop system* (5) *with a guaranteed upper bound $\sqrt{\gamma}$ on its $\mathcal{H}_\infty$ performance.*

$$Q_2(\alpha,\Psi(\alpha))'\begin{bmatrix}-P(\alpha) & 0 & 0\\ 0 & P(\alpha+\Delta\alpha) & 0\\ 0 & 0 & -I\end{bmatrix}Q_2(\alpha,\Psi(\alpha))+\mathrm{He}\left\{\begin{bmatrix}X_1(\alpha)\\X_2(\alpha)\\X_3(\alpha)\end{bmatrix}\begin{bmatrix}\Upsilon(\alpha)\tilde{C}_y(\alpha) & \Upsilon(\alpha)\tilde{D}_y(\alpha) & -I\end{bmatrix}\right\}\prec 0 \tag{10a}$$

$$\begin{bmatrix}W(\alpha) & \mathcal{C}_\Psi(\alpha) & \mathcal{D}_\Psi(\alpha) & \tilde{D}_u(\alpha)\\ \star & P(\alpha) & 0 & 0\\ \star & \star & I & 0\\ \star & \star & \star & 0\end{bmatrix}+\mathrm{He}\left\{\begin{bmatrix}X_4(\alpha)\\X_5(\alpha)\\X_6(\alpha)\\X_7(\alpha)\end{bmatrix}\begin{bmatrix}0 & \Upsilon(\alpha)\tilde{C}_y(\alpha) & \Upsilon(\alpha)\tilde{D}_y(\alpha) & -I\end{bmatrix}\right\}\succ 0 \tag{10b}$$

$$Q_\infty(\alpha,\Psi(\alpha))'\begin{bmatrix}-P(\alpha) & 0 & 0 & 0\\ 0 & P(\alpha+\Delta\alpha) & 0 & 0\\ 0 & 0 & -\gamma I & 0\\ 0 & 0 & 0 & I\end{bmatrix}Q_\infty(\alpha,\Psi(\alpha))+\mathrm{He}\left\{\begin{bmatrix}X_1(\alpha)\\X_2(\alpha)\\X_3(\alpha)\end{bmatrix}\begin{bmatrix}\Upsilon(\alpha)\tilde{C}_y(\alpha) & \Upsilon(\alpha)\tilde{D}_y(\alpha) & -I\end{bmatrix}\right\}\prec 0 \tag{11}$$

$$Q_2(\alpha,\Psi(\alpha))'\begin{bmatrix}-P(\alpha) & 0 & 0\\ 0 & P(\alpha+\Delta\alpha) & 0\\ 0 & 0 & -I\end{bmatrix}Q_2(\alpha,\Psi(\alpha))+\mathrm{He}\left\{\begin{bmatrix}0\\0\\I\end{bmatrix}\begin{bmatrix}Z(\alpha)\tilde{C}_y(\alpha) & Z(\alpha)\tilde{D}_y(\alpha) & -Y(\alpha)\end{bmatrix}\right\}\prec 0 \tag{14a}$$

$$\begin{bmatrix}W(\alpha) & \mathcal{C}_\Psi(\alpha) & \mathcal{D}_\Psi(\alpha) & \tilde{D}_u(\alpha)\\ \star & P(\alpha) & 0 & 0\\ \star & \star & I & 0\\ \star & \star & \star & 0\end{bmatrix}+\mathrm{He}\left\{\begin{bmatrix}0\\0\\0\\I\end{bmatrix}\begin{bmatrix}0 & -Z(\alpha)\tilde{C}_y(\alpha) & -Z(\alpha)\tilde{D}_y(\alpha) & Y(\alpha)\end{bmatrix}\right\}\succ 0 \tag{14b}$$

$$Q_\infty(\alpha,\Psi(\alpha))'\begin{bmatrix}-P(\alpha) & 0 & 0 & 0\\ 0 & P(\alpha+\Delta\alpha) & 0 & 0\\ 0 & 0 & -\gamma I & 0\\ 0 & 0 & 0 & I\end{bmatrix}Q_\infty(\alpha,\Psi(\alpha))+\mathrm{He}\left\{\begin{bmatrix}0\\0\\I\end{bmatrix}\begin{bmatrix}Z(\alpha)\tilde{C}_y(\alpha) & Z(\alpha)\tilde{D}_y(\alpha) & -Y(\alpha)\end{bmatrix}\right\}\prec 0 \tag{17}$$

Proof. The proof is constructed by following the lines of the proof presented in [21] for LTI systems. □

Theorem 3 (Theorem 4) provides parameter-dependent sufficient LMIs for the synthesis of fixed-order LPV controllers (3) for the LPV system (1), such that a closed-loop $\mathcal{H}_2$ ($\mathcal{H}_\infty$) performance is guaranteed.

The convex parameter-dependent synthesis conditions (14) and (17) feature additional conservatism compared to the analysis conditions (10) and (11), respectively. Namely, structural constraints need to be imposed on the slack variables $X_j(\alpha)$ in (10) and (11), as described in the following remark.

Remark 1. *The derivation of the synthesis conditions* (14) *relies on the specific selections $X_j(\alpha)=0$ for $j=1,2,4,5,6$, $X_3(\alpha)=Y(\alpha)$ and $X_7(\alpha)=-Y(\alpha)$ in the analysis conditions* (10)*. Similarly, $X_j(\alpha)=0$ for $j=1,2$ and $X_3(\alpha)=Y(\alpha)$ are selected in the analysis condition* (11) *to arrive at* (17)*. Using* (9) *and imposing the specific structure* (13) *on $Y(\alpha)$, these particular choices allow the reconstruction of a fixed-order LPV controller through the nonlinear transformation* (16)*.*

By making the specific choices as discussed in Remark 1, feasibility of the parameter-dependent LMIs (14) (LMI (17)) for all $(\alpha,\Delta\alpha)\in\Omega$ implies that the closed-loop system H_Ψ is exponentially stable and satisfies the performance bound $\|H_\Psi\|_2^2<\mu$ ($\|H_\Psi\|_\infty^2<\gamma$), as can be seen from application of the projection lemma. This ties in with selecting a full-order controller $\Psi(\alpha)$ with the *desired* closed-loop performance for the computation of a fixed-order LPV controller with similar (desired) closed-loop performance.

It is emphasized that all the optimization variables in (14) and (17) are allowed to be parameter-dependent. Assuming polynomial parameterizations of all optimization variables generally leads to a reduced-order controller with a rational parameter dependency, as implied by (16). In this case, note that the polynomial degree of the numerator and denominator can be prefixed by selecting specific parameterizations for the variables $Y(\alpha)$ and $\bar{\Theta}(\alpha)$. For instance, a fixed-order LPV controller with a polynomial parameter dependency is enforced when $Y_{11}(\alpha)$, $Y_{13}(\alpha)$, $Y_{31}(\alpha)$ and $Y_{33}(\alpha)$ are taken constant (i.e., independent of α).

The $\mathcal{H}_2$ and $\mathcal{H}_\infty$ synthesis conditions can be extended to handle multiple design objectives, by choosing the optimization variables in (16) identical for each performance specification. This choice introduces additional conservatism with respect to single-objective synthesis. However, the remaining variables are chosen differently for each performance specification, since convexity is then retained while keeping conservatism to a minimum.

It is important to stress that the parameter-dependent LMI conditions (14) and (17) are *semi-infinite*, i.e., they should hold for all parameter trajectories $\alpha\in\mathcal{T}$, yielding an infinite number of constraints. A finite set of sufficient LMIs is derived by exploiting the structure of the convex polytopic domain Ω, as described next.

C. LMI relaxations

The fact that the parameter-dependent synthesis LMIs (14) and (17) should hold for all $(\alpha, \Delta\alpha) \in \Omega$ gives rise to semi-infinite optimization problems, which are numerically intractable. To relieve this issue, so-called LMI relaxations are applied to derive a finite set of LMIs whose feasibility guarantees that the parameter-dependent LMIs (14) and (17) hold for all $(\alpha, \Delta\alpha) \in \Omega$.

The chosen relaxation technique, which is closely related to the approach presented in [8], [30], exploits the convex polytopic structure of the parameter-domain Ω (see (2)), and relies on polynomial parameter dependencies and Pólya's theorem [31]–[33]. In contrast to the approach [8], [30], where each point $(\alpha, \Delta\alpha) \in \Omega$ is expressed as a convex combination of *all* the vertices of Ω, we propose to apply a simplicial subdivision of the domain and subsequently use Pólya's theorem to derive a finite set of sufficient LMIs. Namely, for our application, we noticed that such a subdivision results in a significant reduction of numerical complexity while the level of conservatism is maintained. See [34] (p. 65-69) for the technical details.

IV. Application: Lab-Scale Overhead Crane

This section considers the design of a fixed-order multi-objective $\mathcal{H}_2/\mathcal{H}_\infty$ LPV controller for a lab scale overhead crane with varying cable length. First, a description of the overhead crane model and the control objective are provided. Then, the approach from Section III is applied to design a fixed-order LPV controller, followed by experimental validations. The LMIs are implemented and solved in MATLAB using the software packages Yalmip [35] and SeDuMi [36].

A. Overhead crane model

The system under consideration (shown in Fig. 1) consists of a velocity controlled cart on a rail, to which a load is attached through a cable with varying length. The horizontal cart and load position are denoted by x_{cart} [m] and x_{load} [m], respectively, while α [m] defines the cable length and θ [rad] is the swing angle. The system input is a voltage $u \in [-10, 10]$ [V], which scales to cart velocity through a high bandwidth velocity controller. The quantities x_{cart} and θ, as well as the varying cable length α, are measured in real-time. To account for disturbance rejection in the control objective, an additional input d_θ is defined, modeling the effect of an initial swing angle disturbance. Specifically, selecting d_θ as the unit impulse function corresponds to an initial swing angle of 0.1 rad and a horizontal load velocity of 0 m/s. A multiple-input multiple-output 4^{th} order LPV model with an affine dependency on $\alpha \in \mathbb{R}_+$ and a sampling period of 0.01 s is identified using the SMILE technique [10], [37], and represented in state-space form as

$$G : \begin{cases} x(k+1) &= A(\alpha(k))x(k) + B(\alpha(k)) \begin{bmatrix} u(k) \\ d_\theta(k) \end{bmatrix}, \\ \begin{bmatrix} x_{\text{cart}}(k) \\ \theta(k) \end{bmatrix} &= C(\alpha(k))x(k) + D \begin{bmatrix} u(k) \\ d_\theta(k) \end{bmatrix}. \end{cases} \tag{18}$$

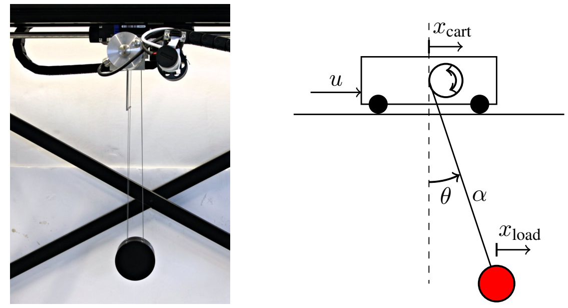


Fig. 1. The overhead crane setup (left) and its schematic representation (right).

We select the set of admissible parameter trajectories as in (2), where Ω is the convex hull of

$$\left\{ \begin{bmatrix} \alpha_L \\ 0 \end{bmatrix}, \begin{bmatrix} \alpha_L \\ b \end{bmatrix}, \begin{bmatrix} \alpha_U - b \\ b \end{bmatrix}, \begin{bmatrix} \alpha_U \\ 0 \end{bmatrix}, \begin{bmatrix} \alpha_U \\ -b \end{bmatrix}, \begin{bmatrix} \alpha_L + b \\ -b \end{bmatrix} \right\} \tag{19}$$

with $\alpha_L = 0.35$ m, $\alpha_U = 0.75$ m, and $b = 0.004$ m/0.01s, corresponding to a cable length varying between 0.35 m and 0.75 m, and a maximum cable hoisting velocity of 0.4 m/s.

B. Control design objective

The aim is to design a fixed-order LPV controller of the form (3) for the identified LPV model (18), achieving a good trade-off between reference tracking of the load position and rejection of swing angle disturbances under the influence of a varying cable length.

We define a reference signal r for the horizontal cart position, and a corresponding error signal $e := r - x_{\text{cart}}$. Note that $x_{\text{load}} \approx r$ whenever $x_{\text{cart}} \approx r$ and $\theta \approx 0$. The controller input is selected as $y := \begin{bmatrix} e & \theta \end{bmatrix}'$. To assure a high bandwidth and good reference tracking, we consider a weighting function described by the continuous-time transfer function

$$W(s) := \frac{s/A_\infty + \omega_c}{s + A_0 \omega_c}, \tag{20}$$

where ω_c [rad/s] is the crossover frequency, while $\lim_{s\to 0} W(s) = 1/A_0$ and $\lim_{s\to\infty} W(s) = 1/A_\infty$. Selecting $\omega_c = 0.2$, $A_0 = -60$ dB and $A_\infty = 100$ dB in (20), this transfer function is discretized using a zero-order hold, resulting in a discrete-time LTI model $W_e : e \to z_e$. Fig. 2 provides a schematic overview of the interconnected system. An $\mathcal{H}_\infty$ performance specification is selected for the channel $r \to z_e$ to assure a high bandwidth, while an $\mathcal{H}_2$ performance specification is imposed on the channel $d_\theta \to \theta$ to account for the rejection of swing angle disturbances. The $\mathcal{H}_2$ norm bound indirectly accounts for a penalty on the control effort. Our choice for $\mathcal{H}_2$ performance stems from the fact that, in this case, minimization of the $\mathcal{H}_2$ norm relates to minimization of the energy in the autonomous response to an initial swing angle deviation.

C. Full-order $\mathcal{H}_2/\mathcal{H}_\infty$ LPV controller design

In order to design a fixed-order LPV controller for the practical LPV model (18), we design a full-order polynomially parameter-dependent LPV controller satisfying the desired

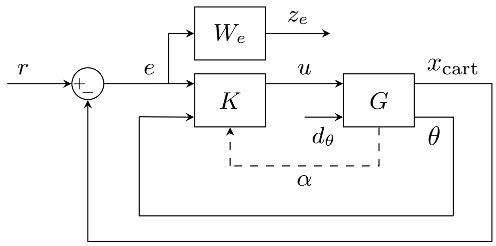


Fig. 2. Block diagram of the closed-loop system corresponding to the overhead crane model (18) interconnected with a dynamic output feedback LPV controller (3).

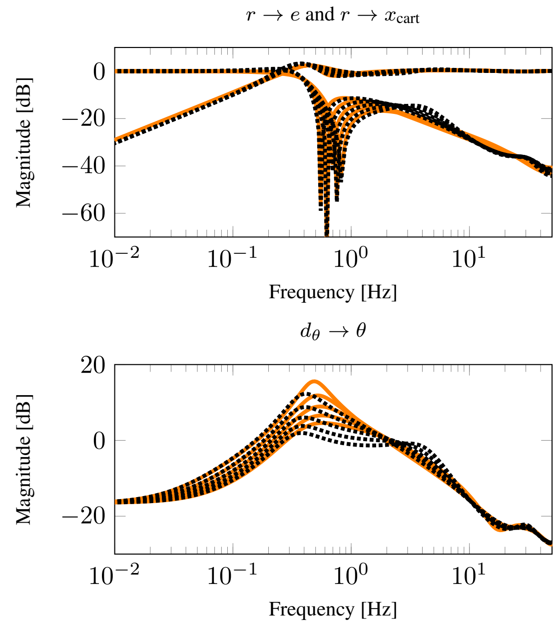


Fig. 3. Bode magnitude plots of the closed-loop system corresponding to the fixed-order LPV controller (dotted black) versus the full-order LPV controller (solid orange), evaluated for 5 equidistant cable lengths $\alpha \in [0.35, 0.75]$.

trade-off between bandwidth and disturbance rejection. This full-order controller is subsequently substituted for $\Psi(\alpha)$ in the synthesis conditions (14) and (17) to design a fixed-order LPV controller in the next subsection.

We consider the design of a strictly proper (i.e., $D_c(\alpha) = 0$) full-order ($n_x = 5$) LPV controller. Following the lines in [8] (see also [38]), parameter-dependent sufficient LMIs for full-order $\mathcal{H}_2/\mathcal{H}_\infty$ LPV controller synthesis result, relying on a well-known nonlinear change of controller variables [39]. For these parameter-dependent LMIs, a finite set of sufficient LMIs is derived as discussed in Subsection III-C. Selecting an affine parameterization of the Lyapunov matrix, the $\mathcal{H}_\infty$ performance bound γ is minimized subject to a prefixed bound μ on the $\mathcal{H}_2$ performance. A lower value of μ guarantees better swing angle disturbance rejection at the expense of a lower bandwidth and a higher γ value, and vice-versa. Controllers with different trade-offs have been implemented and tested experimentally. Based on the experimental responses on a reference step r and an impulse disturbance d_θ, we select the full-order controller corresponding to $\mu = 0.6$ with an $\mathcal{H}_\infty$ bound $\gamma = 0.123$. The state-space model (3) of this controller is characterized by 105 scalar variables.

The reason for selecting an affine parameterization of the Lyapunov matrix stems from the following facts. Firstly, a higher polynomial degree p of the Lyapunov matrix does not result in significantly lower bounds γ for the desired trade-off ($\mu = 0.6$). Secondly, since the degree p is directly related to the degree of the resulting full-order LPV controller (being equal to $p + 1$), a low degree p is desired to keep the LMI problems corresponding to fixed-order LPV synthesis numerically attractive.

D. *Fixed-order $\mathcal{H}_2/\mathcal{H}_\infty$ LPV controller design*

Now we apply the approach of Section III to design a fixed-order LPV controller for the practical LPV model (18), starting from the full-order $\mathcal{H}_2/\mathcal{H}_\infty$ LPV controller with the optimal performance trade-off computed in Subsection IV-C.

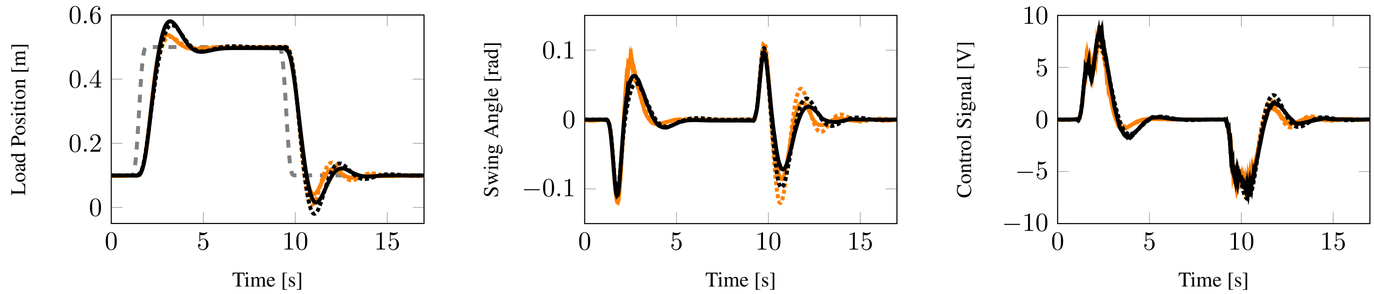
In the parameter-dependent LMIs (14) and (17), we exploit the freedom to select different optimization variables for each performance specification, as explained in Subsection III-B. An affine Lyapunov matrix is selected for both the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance, and the matrix $A_{22}(\alpha)$ is set to zero. Furthermore, matrix $\Psi(\alpha) \in \mathbb{R}^{(n_x+n_u)\times(n_x+n_y)}$ is constructed from the computed full-order LPV controller with the desired performance trade-off. The number of states is selected as $q = 2$, and a polynomial parameterization of degree 2 is considered, corresponding to a fixed-order controller characterized by 36 scalar variables. To this end, $Y_{11}(\alpha)$, $Y_{13}(\alpha)$, $Y_{31}(\alpha)$, $Y_{33}(\alpha)$ are chosen constant, while a polynomial dependency on α of degree 2 is selected for $\bar{\Theta}(\alpha)$. The bandwidth is optimized while the same $\mathcal{H}_2$ performance bound as for the full-order design ($\mu = 0.6$) is maintained.

Numerical issues occur when solving the synthesis LMIs, due to badly conditioned inverses, resulting in unreliable closed-loop performance bounds. Therefore, a reliable performance bound $\gamma = 29.1$ is computed a posteriori by solving analysis LMIs. Note that the obtained performance bound is considerably higher compared to the full-order design. However, solving the analysis LMIs for $b = 0$ results in a bound $\gamma = 0.195$, indicating that the fixed-order LPV controller yields similar performance as the full-order controller for slow parameter variations.

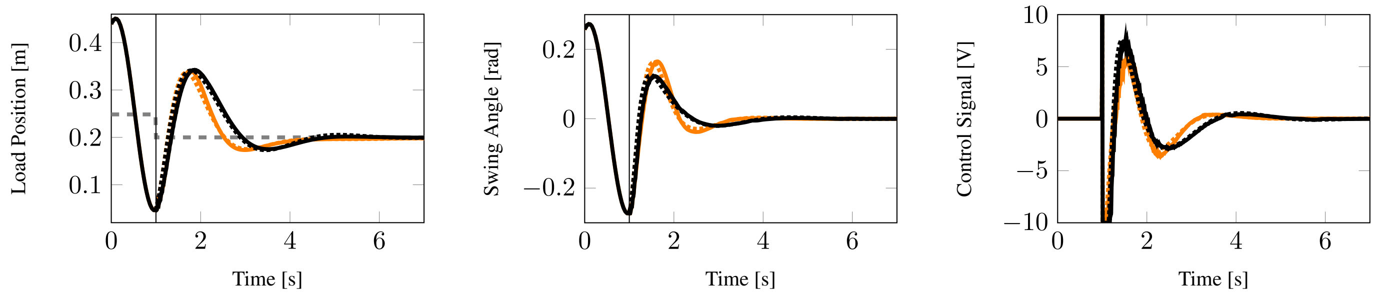
Fig. 3 shows Bode magnitude plots of the closed-loop system corresponding to the fixed-order LPV controller versus the full-order LPV controller, evaluated for 5 equidistant cable lengths α in the interval $[0.35, 0.75]$. In the next subsection, we compare the experimental performances of the fixed- and full-order LPV controller for *varying* cable lengths.

E. *Experimental validation*

Now we discuss the experimental validation of the fixed-order LPV controller design approach from Section III on our lab-scale overhead crane setup described in Subsection IV-A. The closed-loop performance of the full-order and the fixed-order controllers, computed in Subsection IV-C and Subsection IV-D, respectively, is investigated under the influence of a varying cable length. To this end, the following two experiments are performed (see Fig. 4):



(a) Response corresponding to the reference tracking experiment.



(b) Response corresponding to the disturbance rejection experiment.

Fig. 4. Performance of the fixed-order (black) and the full-order (orange) LPV controller. The experimental (solid) and simulated (dashed) responses to a reference trajectory (dashed gray) for the horizontal cart position are shown for both the reference tracking (top) and disturbance rejection (bottom) experiment.

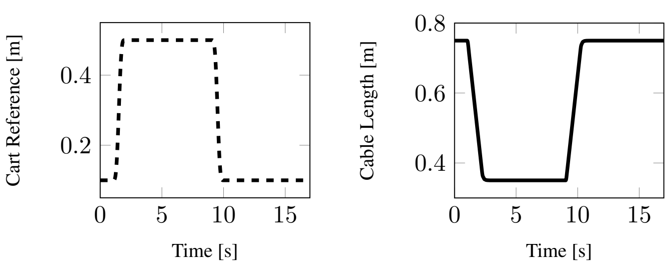


Fig. 5. Reference for the cart position (left, dashed) and the cable length trajectory (right, solid) corresponding to the reference tracking experiment.

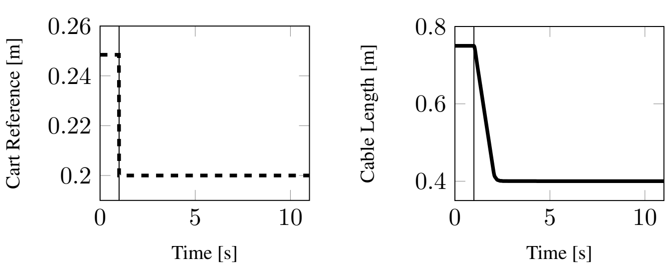


Fig. 6. Reference for the cart position (left, dashed) and the cable length trajectory (right, solid) corresponding to the disturbance rejection experiment. The controller is activated at time 1 second, indicated by the thin vertical black line.

- **Reference tracking:** A reference trajectory corresponding to a back and forth motion with a displacement of 0.4 m is applied for the horizontal cart position in closed loop. This reference trajectory consists of piecewise 9^{th} order polynomials (instead of discontinuous step functions) to avoid actuator saturation. Whenever a change in reference occurs, the cable is hoisted linearly at a rate of 0.4 m/s (see Fig. 5), which is the maximum rate of parameter variation the system can handle.
- **Disturbance rejection:** This experiment consists of three phases (see Fig. 6).
  1) Starting from a system that is initially at rest, the horizontal cart position is changed rapidly without activating the controller, causing a freely swinging load.
  2) The load is freely swinging, while the horizontal cart position is constant.
  3) When the deviation of the load reaches a maximum amplitude, and hence the load velocity is zero, the controller is activated. A fixed reference for the cart position is set, while the load is hoisted linearly.

We observed that the experimental performance of the full-order LPV controller is as desired. Moreover, the fixed-order LPV controller has similar closed-loop performance compared to the full-order controller, as expected from Fig. 3.

Fig. 4a and Fig. 4b confirm that the fixed-order LPV controller yields desired closed-loop behavior. Fig. 4a shows the experimental (solid) and simulated (dotted) response of the full-order (orange) and fixed-order (black) controller to a smooth reference trajectory (dashed gray) for the horizontal cart position and a cable length trajectory as in Fig. 5. Simulated and experimental responses to an initial swing angle disturbance are shown in Fig. 4b for both controllers. In these figures, the controller is activated at time 1 second, corresponding to the cable length trajectory in Fig. 6. Compared to the responses of the full-order controller, the fixed-order controller exhibits a slightly slower response, and more (less) overshoot for the reference tracking (disturbance rejection) experiment.

Since the fixed-order LPV controller is described with only 36 scalar variables compared to 105 for the full-order LPV controller, this modest difference in performance is impressive.

V. Conclusion

An LMI framework to design fixed-order multi-objective $\mathcal{H}_2/\mathcal{H}_\infty$ controllers for discrete-time LPV systems has been presented. Starting from an a priori computed full-order LPV controller stabilizing the LPV system for all possible parameter trajectories, sufficient LMIs for fixed-order $\mathcal{H}_2/\mathcal{H}_\infty$ LPV control design were derived. It has been shown that a multi-objective full-order LPV controller with the desired closed-loop performance can be used as an intuitive starting point for the computation of fixed-order multi-objective LPV controllers. Experimental validations on a lab-scale overhead crane, including thorough comparisons between a full-order and a fixed-order LPV controller, confirmed the viability of the LPV controller design approach for realistic engineering problems.

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